Let $R$ be a $B^*$-algebra with an identity element $e$. Let $E$ be the set of all self-adjoint elements of $R$, and let $D$ ($D_0$) be the set of all elements of $E$ with non-negative (positive) spectra. Let $P$ be the set of real-valued linear functionals on $E$ which are non-negative on $D$, and let $\mathfrak{P}$ be the set of positive functionals on $R$ (i.e., the set of $f$ such that $f(x^*x) \geq 0$ for all $x \in R$). Then $D$ and $D_0$ are convex cones and $D_0$ is the interior of $D$. If $H$ is a closed subspace of $E$ such that $H \cap D$ is empty, then there is a nonzero $f \in P$ such that $f(H) = 0$. There is a nonzero element in $\mathfrak{P}$ if and only if $x^*x + y^*y + \cdots + z^*z = -e$ is impossible. The following conditions are equivalent: $R$ is a $C^*$-algebra; $x^*x + y^*y + \cdots + z^*z = 0$ implies $x = y = \cdots = z = 0$; $P = \mathfrak{P}$ (in the obvious sense). [Reviewer’s remarks: That $D$ is a convex cone has been proved independently by Kelley and Vaught [Theorem 4.7 of the paper reviewed above]. As Kaplansky has observed, the convexity of $D$ has as a consequence an affirmative answer to the conjecture that every $B^*$-algebra is a $C^*$-algebra [Gelfand and Neumark, Mat. Sbornik N.S. 12(54), 197–213 (1943), p. 198; MR0009426 (5,147d)]. The proof is based on the known result that $xy$ and $yx$ always have the same spectrum. In fact, if $-x^*x \in D$ and $x = u - v$ with $u, v \in D$, then $x = 0$ since $2u^2 + 2v^2 - x^*x - xx^* = 0$ and each summand is in the convex cone $D$. It follows from this that $y^*y \in D$ for every $y$ since for $y^*y = u - v$, with $u, v \in D$ and $uv = 0$, we have $(yy^*)yy = -v^3$ and therefore $v = 0$. The case of a $B^*$-algebra without an identity has been handled by Kaplansky by embedding in an algebra with an identity and by Rickart by direct computation (both unpublished).]

Reviewed by J. A. Schatz

{For errata and/or addenda to the original MR item see MR 15,1139 Errata and Addenda in the paper version}