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On a theorem of Gelfand and Neumark and the B^*-algebra.

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Let R be a B^*-algebra with an identity element e. Let E be the set of all self-adjoint elements of R, and let D (D_0) be the set of all elements of E with non-negative (positive) spectra. Let P be the set of real-valued linear functionals on E which are non-negative on D, and let $\mathfrak{P}$ be the set of positive functionals on R (i.e., the set of f such that $f(x^*x) \geq 0$ for all $x \in R$). Then D and D_0 are convex cones and D_0 is the interior of D. If H is a closed subspace of E such that $H \cap D$ is empty, then there is a nonzero $f \in P$ such that $f(H) = 0$. There is a nonzero element in $\mathfrak{P}$ if and only if $x^*x + y^*y + \cdots + z^*z = -e$ is impossible. The following conditions are equivalent: R is a C^*-algebra; $x^*x + y^*y + \cdots + z^*z = 0$ implies $x = y = \cdots = z = 0$; $P = \mathfrak{P}$ (in the obvious sense). [Reviewer's remarks: That D is a convex cone has been proved independently by Kelley and Vaught [Theorem 4.7 of the paper reviewed above]. As Kaplansky has observed, the convexity of D has as a consequence an affirmative answer to the conjecture that every B^*-algebra is a C^*-algebra [Gelfand and Neumark, Mat. Sbornik N.S. **12**(**54**), 197–213 (1943), p. 198; MR0009426 (5,147d)]. The proof is based on the known result that xy and yx always have the same spectrum. In fact, if $-x^*x \in D$ and $x = u - v$ with $u, v \in D$, then $x = 0$ since $2u^2 + 2v^2 - x^*x - xx^* = 0$ and each summand is in the convex cone D. It follows from this that $y^*y \in D$ for every y since for $y^*y = u - v$, with $u, v \in D$ and $uv = 0$, we have $(yv)^*yv = -v^3$ and therefore $v = 0$. The case of a B^*-algebra without an identity has been handled by Kaplansky by embedding in an algebra with an identity and by Rickart by direct computation (both unpublished).]

Reviewed by *J. A. Schatz*

{For errata and/or addenda to the original MR item see MR 15,1139 Errata and Addenda in the paper version}