Analysis of Markov Reward Models with Partial Reward Loss Based on a Time Reverse Approach

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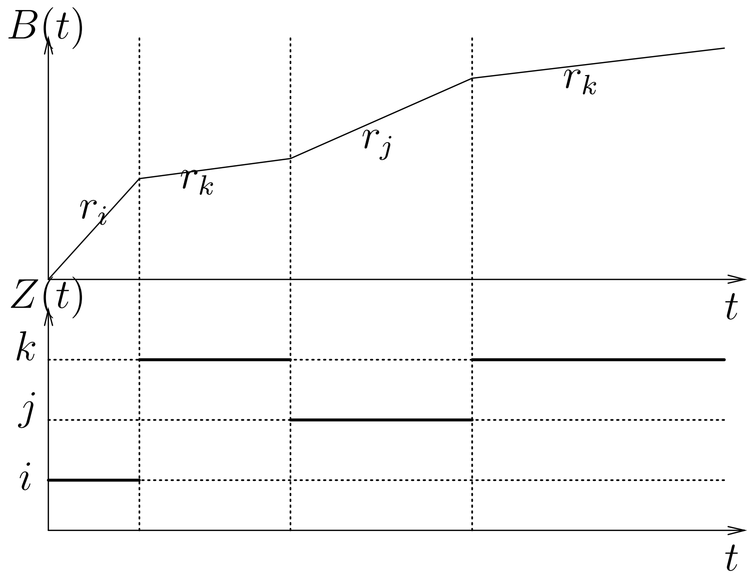
Outline

- Markov Reward models with reward loss
- The difficulty of time forward approach
- The time reverse analysis approach
- Properties of the obtained solution
- Numerical examples
- Conclusions

Markov Reward models without reward loss

Markov reward models (MRM)

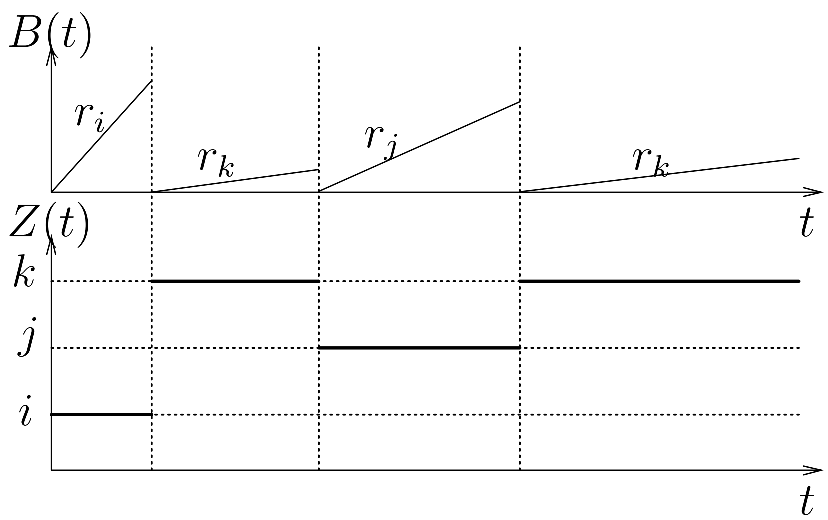
- a finite state CTMC,
- non negative reward rates (r_i),
- performance measures:
  - reward accumulated up to time t,
  - time to accumulate reward w.



Markov Reward models with total reward loss

We consider

- first order MRM (deterministic dependence on $Z(t)$),
- without impulse reward,
- but with potential reward loss at state transition.



Markov Reward models with partial reward lo

In case of partial reward loss:

- α_i remaining portion of reward when leaving state i,
- the lost reward is proportional to:
  - total accumulated reward
    $\Rightarrow$ partial total loss,
  - reward accumulated in the last state
    $\Rightarrow$ partial incremental loss.

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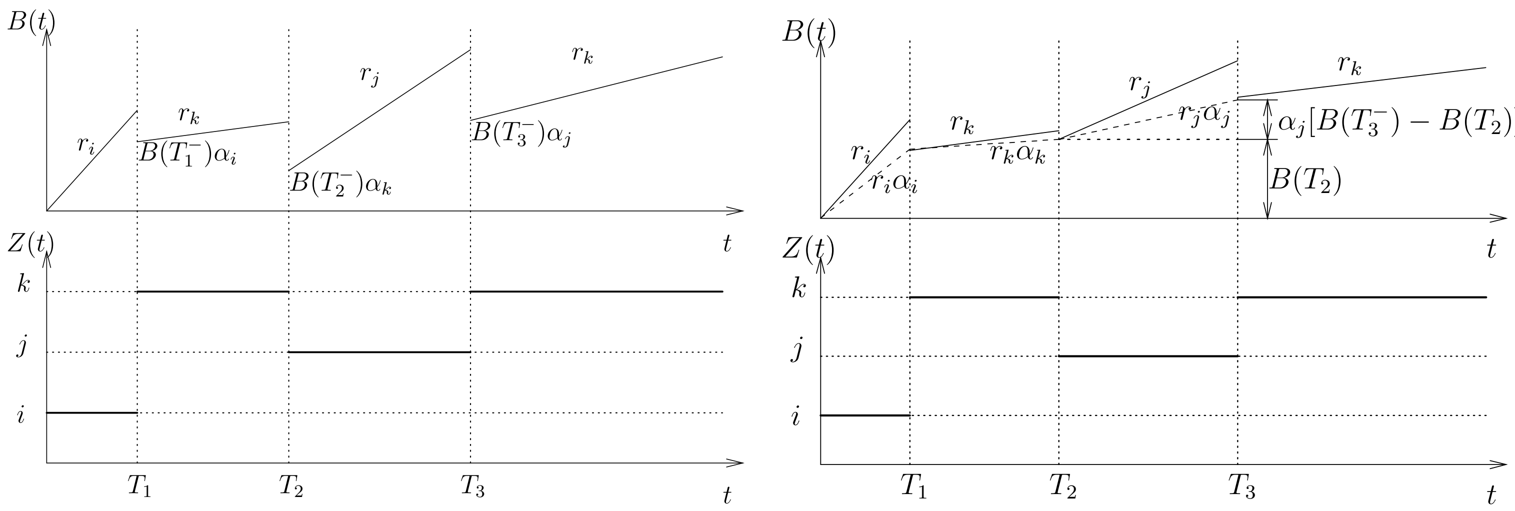
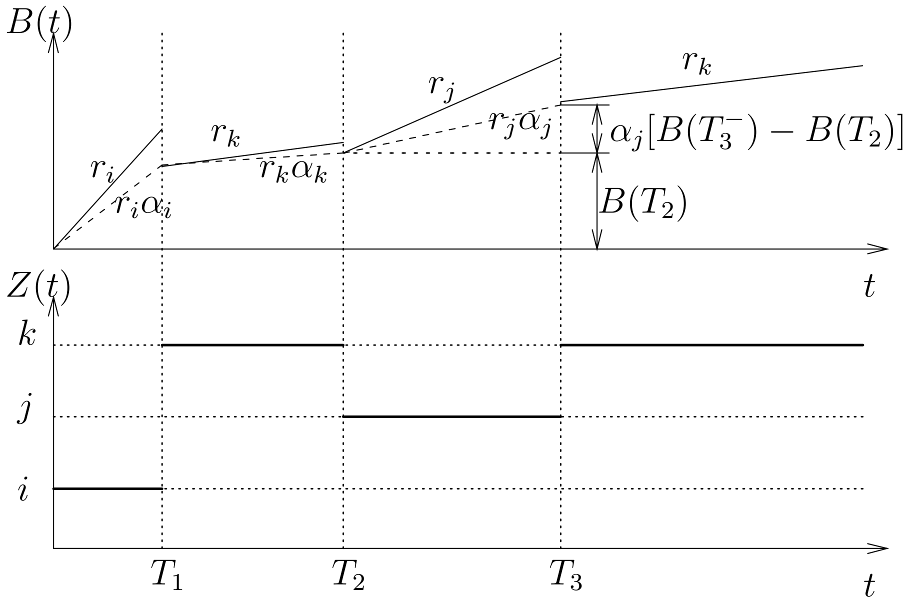


Time forward approach

Possible interpretation:

- Reduced ($r_i \alpha_i$) reward accumulation up to the last state transition,
- and total (r_i) reward accumulation in the last state without reward loss.



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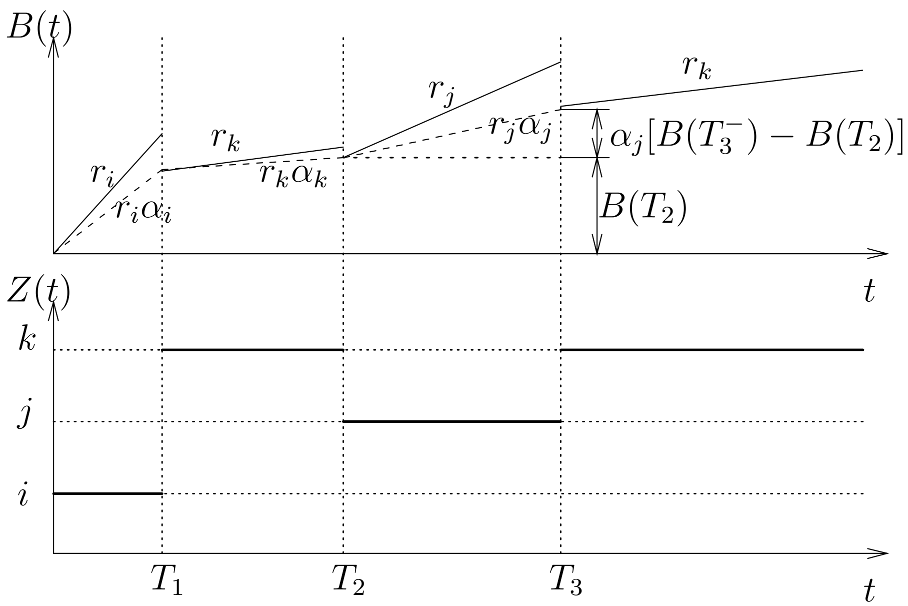

Unfortunately, the last state transition before time T is not a stopping time.

Time reverse approach

Behaviour of the time reverse process:

- Inhomogeneous CTMC

  with initial probability $\overleftarrow{\underline{\gamma}}(0) = \underline{\gamma}(T)$

  and generator $\overleftarrow{\mathbf{Q}}(\tau) = \{\overleftarrow{q_{ij}}(\tau)\}$,

  where

$$\overleftarrow{q_{ij}}(\tau) = \begin{cases} \dfrac{\gamma_j(T-\tau)}{\gamma_i(T-\tau)}\, q_{ji} & \text{if } i \neq j, \\ -\displaystyle\sum_{k \in S, k \neq i} \dfrac{\gamma_k(T-\tau)}{\gamma_i(T-\tau)}\, q_{ki} & \text{if } i = j. \end{cases}$$

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- Total (r_i) reward accumulation in the first state,
- and reduced ($r_i \alpha_i$) reward accumulation in all consecutive states
- without reward loss.

Time reverse approach

Potential model description:

duplicate the state space to describe

- the total reward accumulation in the first state (r_i),
- and the reduced reward accumulation in all further states $(r_i \alpha_i)$.

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$$\pi^*(0) = [\underline{\gamma}(T), \underline{0}], \quad \overleftarrow{\mathbf{Q}}^*(\tau) = \begin{array}{|c|c|} \hline \overleftarrow{\mathbf{Q}_D}(\tau) & \overleftarrow{\mathbf{Q}}(\tau) - \overleftarrow{\mathbf{Q}_D}(\tau) \\ \hline \mathbf{0} & \overleftarrow{\mathbf{Q}}(\tau) \\ \hline \end{array}, \quad \mathbf{R}^* = \begin{array}{|c|c|} \hline \mathbf{R} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{R}_\alpha \\ \hline \end{array}$$

Inhomogeneous differential equation

Introducing

$$\overleftarrow{Y}_i(\tau, w) = Pr(\overleftarrow{B}(\tau) \leq w, \overleftarrow{Z}(\tau) = i)$$

we can apply the analysis approach available for inhomogeneous MRMs.

It is based on the solution of the inhomogeneous partial differential equation

$$\frac{\partial}{\partial \tau} \underline{\overleftarrow{Y}}(\tau, w) + \frac{\partial}{\partial w} \underline{\overleftarrow{Y}}(\tau, w)\mathbf{R} = \underline{\overleftarrow{Y}}(\tau, w)\overleftarrow{\mathbf{Q}}(\tau) \ ,$$

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where $\underline{\overleftarrow{Y}}(\tau, w) = \{\overleftarrow{Y}_i(\tau, w)\}$.

But a drawback of this approach is that it requires the computation of $\overleftarrow{\mathbf{Q}}(\tau)$.

Homogeneous differential equation

To overcome this drawback we introduce the conditional distribution of reward accumulated by the reverse process

$$\overleftarrow{V}_i(\tau, w) = Pr(\overleftarrow{B}(\tau) \leq w \mid \overleftarrow{Z}(\tau) = i)$$

and the row vector $\underline{\overleftarrow{V}}(\tau, w) = \{\overleftarrow{V}_i(\tau, w)\}$.

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Using this performance measure we have to solve

$$\frac{\partial}{\partial \tau}\underline{\overleftarrow{V}}(\tau, w) + \frac{\partial}{\partial w}\underline{\overleftarrow{V}}(\tau, w)\mathbf{R} = \underline{\overleftarrow{V}}(\tau, w)\mathbf{Q}^T,$$

where $\mathbf{Q}^T$ is the transpose of $\mathbf{Q}$.

Block structure of the differential equation

Utilizing the special block structure of the $\mathbf{Q}'(\tau)$ and the $\mathbf{R}'$ matrices (of size $2\#S$) we can obtain two homogeneous partial differential equations of size $\#S$:

$$\frac{\partial}{\partial \tau}\overleftarrow{\underline{X1}}(\tau,w) + \frac{\partial}{\partial w}\overleftarrow{\underline{X1}}(\tau,w)\mathbf{R} = \overleftarrow{\underline{X1}}(\tau,w)\mathbf{Q}_D \ ,$$

and

$$\frac{\partial}{\partial \tau}\overleftarrow{\underline{X2}}(\tau,w)+\frac{\partial}{\partial w}\overleftarrow{\underline{X2}}(\tau,w)\mathbf{R}_\alpha = \overleftarrow{\underline{X1}}(\tau,w)(\mathbf{Q}-\mathbf{Q}_D)^T + \overleftarrow{\underline{X2}}(\tau,w)\mathbf{Q}^T \ ,$$

Moments of accumulated reward

The analysis approach available for inhomogeneous MRMs allows to describe the moments of IMRMs with an inhomogeneous ordinary differential equation.

Similar to the reward distribution case, this approach is also applicable for our model, but it requires the the computation of $\overleftarrow{\mathbf{Q}}(\tau)$.

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Using similar state dependent moment measures we obtain homogeneous ordinary differential equations

$$\frac{d}{d\tau}\overleftarrow{\underline{M1}}^{(n)}(\tau) = n\overleftarrow{\underline{M1}}^{(n-1)}(\tau)\mathbf{R} + \overleftarrow{\underline{M1}}^{(n)}(\tau)\mathbf{Q}_D \ ,$$

and

$$\frac{d}{d\tau}\overleftarrow{\underline{M2}}^{(n)}(\tau) = n\overleftarrow{\underline{M2}}^{(n-1)}(\tau)\mathbf{R}_\alpha + \overleftarrow{\underline{M1}}^{(n)}(\tau)(\mathbf{Q}-\mathbf{Q}_D)^T + \overleftarrow{\underline{M2}}^{(n)}(\tau)\mathbf{Q}$$

Randomization based numerical method

The ordinary differential equation with constant coefficients allows to compose a randomization based numerical method.

$$\underline{\overleftarrow{M1}}^{(n)}(\tau) = \tau^n \, \underline{e} \, \mathbf{R}^n \mathbf{E}_D(\tau) \ ,$$

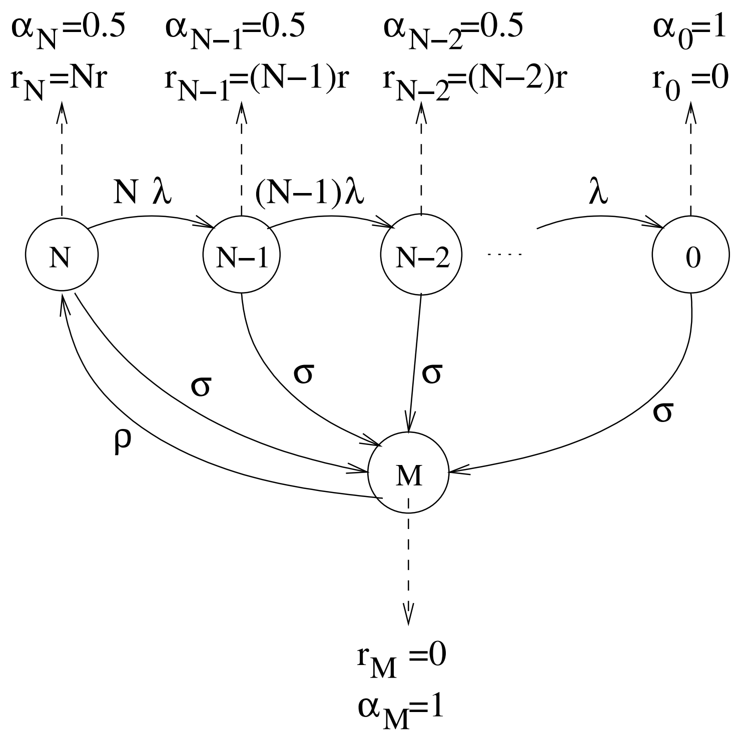
and

$$\underline{\overleftarrow{M2}}^{(n)}(\tau) = n! d^n \sum_{k=0}^{\infty} e^{-\lambda\tau} \, \frac{(\lambda\tau)^k}{k!} \, \underline{D}^{(n)}(k),$$

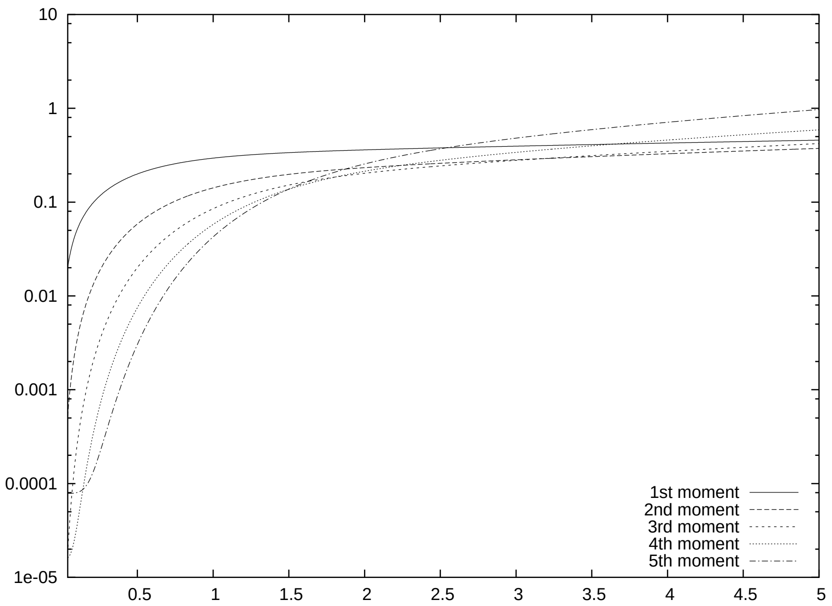
where

$$\underline{D}^{(n)}(k) = \begin{cases} \underline{e}\,(\mathbf{I} - \mathbf{A}_D^k) & n = 0 \\ 0 & k \leq n, n \geq 1 \\ \underline{D}^{(n-1)}(k-1)\mathbf{S}_\alpha + \underline{D}^{(n)}(k-1)\mathbf{A} + & \\ \quad \binom{k-1}{n} \, \underline{e} \, \mathbf{S}^n \mathbf{A}_D^{k-1-n} (\mathbf{A} - \mathbf{A}_D) & k > n, n \geq 1 \end{cases}$$

Numerical Example



Structure of the Markov chain



Moments of the accumulated reward

With parameters $N = 500000$, $\lambda = 0.000004$, $\sigma = 1.5$, $\rho = 0.1$, $r = 0.000002$, $\alpha = 0.5$,

Conclusions

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- partial differential equation $\Rightarrow$ ordinary differential equations,

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We propose an analysis method with the following features:

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- inhomogeneous differential equation $\Rightarrow$ proper performance measure,
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- numerical stability, error control $\Rightarrow$ randomization based analysis.

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Thanks for your attention.