Bounding the Mean Cumulated Reward up to Absorption

A. P. Couto and G. Rubino

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Motivations

- paper area: evaluating performance, dependability and performability measures using Markov models
- models considered: no closed-form solutions, large state spaces
- problem: no exact (numerically speaking) solution available
- goal: obtain approximations of standard metrics with error bounds, avoiding the large state spaces
- starting point: in usual dependability contexts, we have stiffness
Main ideas

• We compute lower and upper bound of the Expected Cumulated Reward up to Absorption (ECRA),
• for a system modeled by an absorbing Markov chain with states weighted by rewards.
• The approach merges the idea of state space reduction using stochastic complementation with a path-based procedure.
• The idea of fast and slow transitions and states is used.
2/ The ECRA metric

Markov model

- state space: \( \{1, \cdots, M\} \cup \{M + 1\} \), with \( M \gg 1 \)
- CTMC \( X \) where \( 1, \cdots, M \) are transient and \( M + 1 \) is absorbing; \( \Omega = \{1, \cdots, M\} \)
- infinitesimal generator of \( X \) and its initial distribution are decomposed with respect to \((\Omega, \{M + 1\})\):
  \[
  \begin{pmatrix}
  Q & q \\
  0 & 0
  \end{pmatrix}, \quad q = -Q 1; \quad \begin{pmatrix}
  \alpha \\
  0
  \end{pmatrix}
  \]
- from any transient state \( i \) there is a path to state \( M + 1 \)
Adding rewards

- \( r_i \): reward rate at state \( i \); \( r_{M+1} = 0 \).
- The Cumulated Reward until Absorption is the random variable
  \[
  R^X_{\infty} = \int_0^\infty r_X(t) \, dt.
  \]
- Target: \( E(R^X_{\infty}) = \text{ECRA} \).
The canonical embedded DTMC is decomposed on \((\Omega, \{M + 1\})\):

\[
\begin{pmatrix}
P & p \\
0 & 1
\end{pmatrix}, \quad P_{ij} = Q_{ij}/(-Q_{ii}), \quad p = (I - P)1
\]

Let \(c = (\cdots r_i/q_i \cdots)^T\).

Let \(E(R_X^\infty | X(0) = i) = \rho_i\), and \(\rho = (\cdots \rho_i \cdots)^T\).

Then, \(\rho = (I - P)^{-1}c\) (and ECRA \(\equiv \langle \alpha, \rho \rangle\)).

From now, goal = bounding ECRA through bounding \(\rho\).
Key concept: fast and slow states

- transition \((i, j)\) is fast iff \(Q_{i,j} \geq \theta > 0\)
- transition \((i, j)\) is slow iff \(0 < Q_{i,j} < \theta\)
- state \(i\) is fast iff for some \(j\), transition \((i, j)\) is fast
- state \(i \neq M + 1\) \(^1\) is slow iff it is not fast
- \(S = \{i : i\) is slow\} \\(\ni\) \(X(0)\); \(F = \{i : i\) is fast\}
- Assumption: there is no circuit only composed of fast transitions.
- We can decompose matrices such as \(Q\) or \(P\), and vectors such as \(\alpha\), or \(\rho\), on \(S, F\). For instance,

\[
P = \begin{pmatrix} P_S & P_{SF} \\ P_{FS} & P_F \end{pmatrix}, \quad \rho = \begin{pmatrix} \rho^S \\ \rho^F \end{pmatrix}.
\]

\(^1\)State \(M + 1\) excluded just to simplify the presentation.
Going to discrete time through uniformization

- **Y**: DTMC obtained by uniformizing $X$ w.r.t. $\eta$

  \[
  \text{t.p.m. (on } \Omega, \{M + 1\} \text{): } \begin{pmatrix}
  U & u \\
  0 & 1
  \end{pmatrix}
  \]

  \[
  U = I + Q/\eta, \quad u = (I - U)1 = q/\eta
  \]

- Also, $U = \begin{pmatrix} U_S & U_{SF} \\ U_{FS} & U_F \end{pmatrix}$.

- Define $R^Y_\infty = \sum_{k=0}^{\infty} r^Y_{(k)}$ and $\text{ECRA}^Y = E( R^Y_\infty )$. 
Going to discrete time through uniformization (cont.)

- We have $\text{ECRA}^Y = \langle \alpha , (I - U)^{-1}c \rangle$
- and $\text{ECRA}^X = \frac{\text{ECRA}^Y}{\eta}$

So, from this point, our target is $\text{ECRA}^Y$.

- To simplify, let us rename $\rho_i = E(R^Y_\infty | Y(0) = i), i \in S$, and $\rho$ the corresponding vector.
- We will bound this conditional expected cumulated reward vector $\rho$ (so, working on the DTMC $Y$).
Going to discrete time through uniformization (cont.)

• Write

\[ R_{\infty}^{Y,S} = \sum_{k=0}^{\infty} r_{Y(k)} 1_{Y(k) \in S}, \quad R_{\infty}^{Y,F} = \sum_{k=0}^{\infty} r_{Y(k)} 1_{Y(k) \in F}, \]

then \( \rho_i^S = E(R_{\infty}^{Y,S} \mid Y(0) = i) \), \( \rho_i^F = E(R_{\infty}^{Y,F} \mid Y(0) = i) \).

• We will start from

\[ \rho = \rho^S + \rho^F. \]

• Intuitively, in case of a highly dependable system, \( \rho \approx \rho^S \).
Decomposing the state space

- Let \( \tilde{Y} \) be the DTMC obtained by stochastically complementing the subset \( S \) of slow states (plus state \( M + 1 \)) in \( Y \).
- Process \( \tilde{Y} \) is called the reduction of \( Y \) with respect to \( S \cup \{M + 1\} \). It can be also defined by “sampling” from \( Y \) when \( Y \) visits \( S \cup \{M + 1\} \).
- The t.p.m. of \( \tilde{Y} \) is

\[
\begin{pmatrix}
\tilde{U} & \tilde{u} \\
0 & 1
\end{pmatrix}, \quad \tilde{u} = (I - \tilde{U})1.
\]

where \( \tilde{U} = U_S + U_{SF}(I - U_F)^{-1} U_{FS} \)
Decomposing the state space (cont.)

- Define $R_{\infty}^{\tilde{Y}} = \sum_{k=0}^{\infty} r_{\tilde{Y}(k)}$ and $\text{ECRA}^{\tilde{Y}} = E(R_{\infty}^{\tilde{Y}})$

- For $i \in S$, define $\tilde{\rho} = E(R_{\infty}^{\tilde{Y}} \mid \tilde{Y}(0) = i)$, and denote by $\tilde{\rho}$ the corresponding vector.

- Observe that $\tilde{\rho} = \rho^S$. 
Lower bounding procedure

- Fix some $N \geq 2$. For all $i, j \in S$, we compute the $|S| \times |S|$ matrix $U'_N$ where $(U'_N)_{ij}$ is the probability for $Y$ to go from $i$ to $j$ by remaining in $F$ in between, in no more than $N$ steps.

- For this, we use a BFS (Breadth First Search) with depth $N$, to build all paths of the form

$$\left(i, f_1, f_2, \cdots, f_{N-1}, j, \right),$$

where $f_1, f_2, \cdots, f_{N-1} \in F$.

- Clearly, $U'_N \leq \tilde{U}$.

- If we compute now $\rho' = (I - U'_N)^{-1}c^S$, we have

$$\rho'_i \leq \tilde{\rho}_i = \rho^S_i \leq \rho^S_i + \rho^F_i = \rho_i.$$

- Our lower bound is thus $\rho'$ obtained by solving the linear system with size $|S|$:

$$(I - U'_N)\rho' = c^S.$$ 

**Remark:** if by chance the model allows to compute $\tilde{U}$, then our lower bound of $\rho$ will be simply $\tilde{\rho}$. 
Upper bounding procedure

• Writing $\rho_i = \rho_i^S + \rho_i^F$, we will upper-bound both terms in this representation.
Upper-bound of $\rho^S = \tilde{\rho}$

- We had $U'_N \leq \tilde{U}$; write $U''_N = \tilde{U} - U'_N \geq 0$.

$$U''_N = U_{SF}(U_F^{N-1} + U_F^N + \cdots)U_{FS} = U_{SF}U_F^{N-1}V U_{FS},$$

where $V = (I - U_F)^{-1}$.

$V_{ij}$ is the mean # of visits made by $Y$ to state $j$, starting at $i$, before leaving $F$ ($i, j \in F$).
A bounding Lemma

Assuming there are no circuits in $F$ only composed of fast transitions, the mean sojourn time of $Y$ in $F$ can be upper bounded by

$$\sigma = \frac{\eta}{\mu} \left( \frac{1 - (1 - \psi)^D}{\psi(1 - \psi)^D} \right),$$

where

$$\lambda = \max_{i \in F} \sum_{j: (i, j) \text{ slow}} Q_{ij}, \quad < \mu = \min_{i \in F} \sum_{j: (i, j) \text{ fast}} Q_{ij},$$

$\psi = \lambda / \mu < 1$ and $D$ is the maximal number of fast transitions starting in $F$, before leaving $F$. 
Upper-bound of $\rho^S = \tilde{\rho}$ (cont.)

- Now, we can bound $U''_N$ as follows:
  \[ U''_N \leq B = \sigma U_{SF} U_F^{N-1} 1 U_{FS}. \]
  \(1\) is here a $|F| \times |F|$ matrix composed of ones.
- We have \(\lim_{N \to \infty} B = 0\).

Finally, $\rho^S = \tilde{\rho} = \left( \sum_{k=0}^{\infty} \tilde{U}^k \right) c^S$

\[ = \left( \sum_{k=0}^{\infty} (U'_N + U''_N)^k \right) c^S \]
\[ \leq \left( \sum_{k=0}^{\infty} (U'_N + B)^k \right) c^S = \varphi. \]

See that $\varphi$ is computed again solving a new linear system with size $|S|$ (the system \((I - U'_N - B)\varphi = c^S\)).
Upper-bound of $\rho^F$

- For upper bounding $\rho^F$, consider $\nu_i$, the mean # of visits that $Y$ makes to subset $F$ when starting from $i \in S$; we have

$$\rho_i^F \leq \nu_i \sigma r^*, \quad r^* = \max\{r_k, \ k \in F\}.$$  

- If $W = (I - \tilde{U})^{-1}$, then $\nu_i = \sum_{j \in S} W_{ij} U_{jF}$

- From the preceding results,

$$\rho_i^F \leq \sigma r^* \sum_{j \in S} W_{ij} U_{jF}.$$
Upper-bound of $\rho^F$ (cont.)

- If $u_F$ is the vector indexed on $S$ whose $j$th entry is $U_jF$, then
  $\nu = W u_F$ and
  $$\rho^F \leq \sigma r^* W u_F.$$  

- Now, $W u_F = \sum_{n \geq 0} \tilde{U}^n u_F \leq \sum_{n \geq 0} (U'_N + B)^n u_F = \tau$.

- So, our upper bound of $\rho^F$ is $\sigma r^* \tau$ where $\tau$ is the solution to
  $$(I - U'_N - B) \tau = u_F \text{ (for } N \text{ large enough so that } ||U'_N + B|| < 1).$$

**Remark:** if $\tilde{U}$ is available, then $\nu$ is computed from $(I - \tilde{U}) \nu = u_F$ and the upper bound is $\sigma r^* \nu$. 
A system with identical components

- The system is composed of $m$ identical and independent components.
- Each component evolves among 3 states: operational, down and being repaired, definitely down.
- If $\Phi =$ failure rate, $c =$ covering factor and $\gamma =$ repair rate, we used $c\Phi = 10^{-6}$, $(1 - c)\Phi = 10^{-7}$ and $\gamma = 1$.
- The threshold $\theta$ was $\theta = \gamma$.
- The symmetries of the system allowed us to work on a reduced state space (strong lumpability).
- We used $m = 10, 20, 30, 40, 50$ leading to $66, 231, 496, 861, 1326$ states resp.
- Here, $|S| = m$. 
A system with identical components: some results

<table>
<thead>
<tr>
<th>$m$</th>
<th>Lower Bound</th>
<th>Exact</th>
<th>Upper Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$2.9285 \times 10^7$</td>
<td>$2.9285 \times 10^7$</td>
<td>$2.9285 \times 10^7$</td>
</tr>
<tr>
<td>20</td>
<td>$3.5974 \times 10^7$</td>
<td>$3.5975 \times 10^7$</td>
<td>$3.5975 \times 10^7$</td>
</tr>
<tr>
<td>30</td>
<td>$3.9932 \times 10^7$</td>
<td>$3.9933 \times 10^7$</td>
<td>$3.9933 \times 10^7$</td>
</tr>
<tr>
<td>40</td>
<td>$4.2765 \times 10^7$</td>
<td>$4.2765 \times 10^7$</td>
<td>$4.2767 \times 10^7$</td>
</tr>
<tr>
<td>50</td>
<td>$4.4959 \times 10^7$</td>
<td>$4.4959 \times 10^7$</td>
<td>$4.4961 \times 10^7$</td>
</tr>
</tbody>
</table>
Same system, but without using strong lumpability

Two cases considered:

1. $m = 11$ components $\Rightarrow 177.147$ states; $|S| = 2.048$
2. $m = 12$ components $\Rightarrow 531.441$ states; $|S| = 4.096$

• depth in the BFS: $N = 6$

Numerical values:

<table>
<thead>
<tr>
<th>$m$</th>
<th>Lower Bound</th>
<th>Exact</th>
<th>Upper Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>$3.0116 \times 10^7$</td>
<td>$3.0177 \times 10^7$</td>
<td>$3.0206 \times 10^7$</td>
</tr>
<tr>
<td>12</td>
<td>$3.0925 \times 10^7$</td>
<td>$3.1026 \times 10^7$</td>
<td>$3.1031 \times 10^7$</td>
</tr>
</tbody>
</table>
Same system, but now the components belong to 3 classes ("CPU", "memory", "disks")

- Associated with each class there is a cost when the component is down.
- We bound here the ECRA.
- We don’t use the strong lumpability property.
- Example: with 4 CPU, 5 memory units and 3 disks, we get 531 441 states; $|S| = 12$.
- Depth in the BFS: $N = 6 \Rightarrow$ relative error $\approx 10^{-4}$.
As before, but we replace exponentials by equivalent Coxian distributions

- The exact value is still available,
- and the state space can be huge.
- We don’t use the strong lumpability property.
- An example: in a configuration we get $\approx 430$ M states and $|S| \approx 1.7$ M states.
- Depth in the BFS: $N = 7 \rightsquigarrow$ a relative error $\approx 10^{-3}$. 
Some concluding remarks

• Some refinements under study.
• In cases where $|S| \ll 1$, we are exploring the use of the stochastic complement with respect to a subset $S' \supset S$ plus the absorbing state (some results also in the paper \(^2\)).
• Other linear algebra ideas are also under study, based on previous work done on stationary measures (on irreducible models). This can be of interest in particular in cases where $|S|$ can still be large.

\(^2\)Presented at the “A. A. Markov Anniversary Meeting” (MAM), Charleston, USA, June 12–14 2006.