The Dynamic Analysis and Design of A Communication link with Stationary and Nonstationary Arrivals

–Five dubious ways to dynamically analyze and design a communication system

Wenhong Tian, Harry Perros

Department of Computer Science
North Carolina State University
Raleigh, NC, 27695

June 12, 2006

Outline

1. **Introduction**
   - Andrei Andreevich Markov (1856-1922); Why are we doing this research?
2. **The Nonstationary Erlang Loss System**
   - Stationary Arrivals
   - Time-varying Arrivals
   - Dimensioning A Single Link
3. **Different Solutions to Calculate Blocking Probabilities**
   - The Exact Closed-form Solutions
   - The Ordinary Differential Equations Solver Approach
   - The Truncated Markov Process Approximation
   - The Fixed Point Approximation (FPA) Method
   - The Large Deviation Approach
   - Numerical results
4. **Conclusion**

Outline

In the Honor of A. A. Markov

In this meeting: Life and History of A. A. Markov; applications (five greatest vs. risk etc); past,present and future of his chains; theory and numerical solutions of MC etc.
A good source may be: G. P. Basharin, A.N. Langville and V. A. Naumov, “The life and work of A. A. Markov”, Linear Algebra and its applications. Vol. 386: 3-26, 2004.
“Just because Google uses Markov chain (random walk actually), does not mean MC is already very well understood and applied.”

Problem

Most of performance analysis and network design solutions are based on steady-state analysis (classic). But real traffic is more likely nonstationary (or time-dependent). Nonstationary queueing models are needed for many complex production, service, communication and air transportation systems.
Q: how do we provide nonstationary analysis and design(extension)?

Example: time-of-day effect

"Real life is nonstationary. The number of telephone calls made during 5 minutes interval of 2:07 pm to 2:12 pm on a Wednesday afternoon is considerably larger than the number of calls made during the 5 minute interval of 3:46 am to 3:51 am Monday morning."–W.A. Massey [10]

Solutions

Time-dependent blocking probabilities can be computed

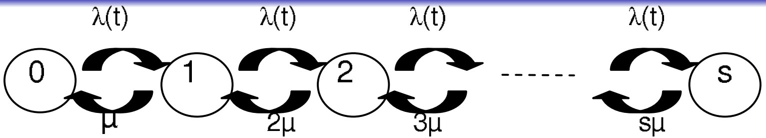
- Time-dependent blocking prob. for constant arrival rates
- Time-dependent blocking prob. for time-varying arrival rates
- Dynamic dimensioning approaches for a single link

Major references

1. Jagerman [6], “Nonstationary blocking in telephone networks”, 1975.
2. C. Moler and C. Van Loan[12], “Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later”, SIAM Rev., 45(1):pp. 3-49, 2003.
3. Mandjes and Ridder [9], “A large deviations approach to the transient of the Erlang loss model”,2001.
4. Nayak and Sivarajan [13], *“ Dimensioning Optical Networks Under Traffic Growth Models”*,2003
5. Karagiannis et al. [7], “A nonstationary Poisson view of Internet traffic”, In *the Proceedings of INFOCOM 2004*

Outline

Fig.0 The Markov chain of Erlang loss model for a single service center



Differential Equations for M/M/s/s System

From Markov chain, The loss system M/M/s/s may be represented by the following set of forward differential equations:

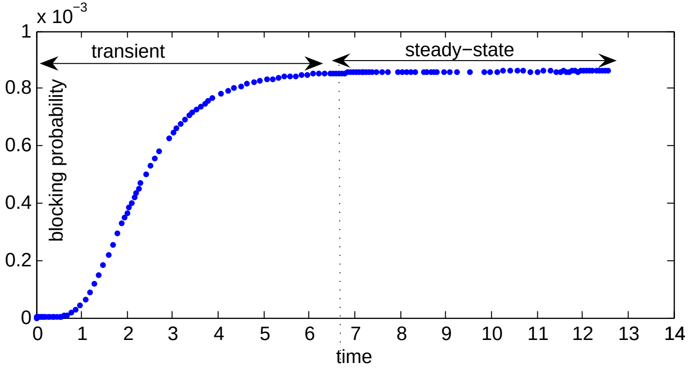
$$P'_0(t) = \mu P_1(t) - \lambda P_0(t) \quad (1)$$

$$P'_n(t) = \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t) - (\lambda + n\mu)P_n(t), 1 \leq n < s, \quad (2)$$

$$P'_s(t) = \lambda P_{s-1}(t) - s\mu P_s(t) \quad (3)$$

where $P_0(t) + P_1(t) + P_2(t) + ... + P_s(t)$=1, and $0 \leq P_n(t) \leq 1$, for $t \geq 0$ and n=$0, 1, 2, ..., s$.

Fig.1 time-dependent blocking probabilities of $M/M/8/8$, $\rho = 2$



Differential Equations for M(t)/M/s/s System

From Markov chain, loss system M(t)/M/s/s may be represented by the following set of forward differential equations:

$$P_0'(t) = \mu P_1(t) - \lambda(t)P_0(t) \tag{4}$$

$$P_n'(t) = \lambda(t)P_{n-1}(t)+(n+1)\mu P_{n+1}(t)-(\lambda(t)+n\mu)P_n(t), 1 \leq n < s, \tag{5}$$

$$P_s'(t) = \lambda(t)P_{s-1}(t) - s\mu P_s(t) \tag{6}$$

where $P_0(t) + P_1(t) + P_2(t) + ... + P_s(t)$=1, $t \geq 0$, and 0 $\leq P_n(t) \leq 1$, for $t \geq 0$ and n=$0, 1, 2, ..., s$.

Fig.2 time-dependent blocking probabilities of the $M(t)/M/220/220$, where $\lambda(t) = 180 + 50sin(2(t + 20))$

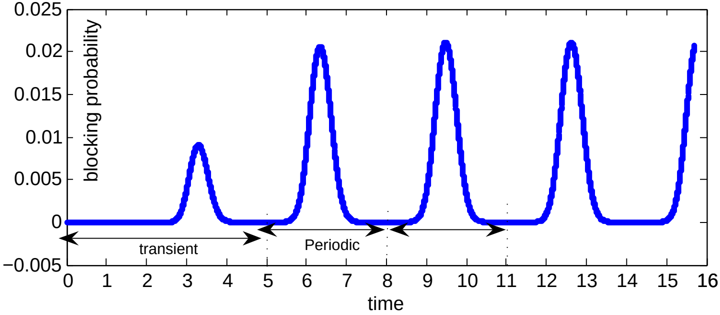


Fig.3 A dimensioning example of a single link with stationary arrival rate λ=10

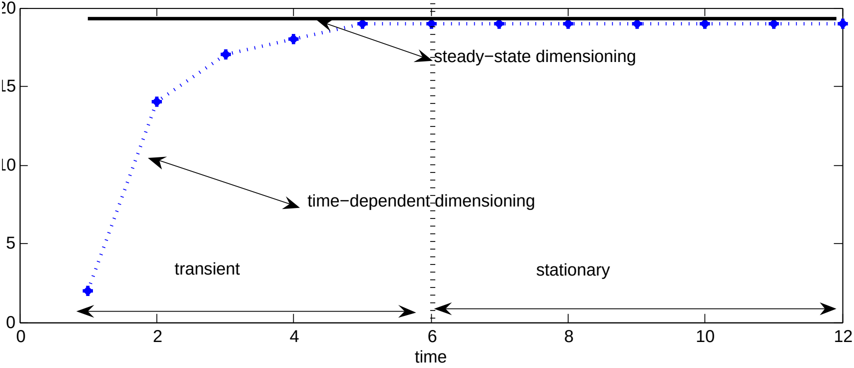
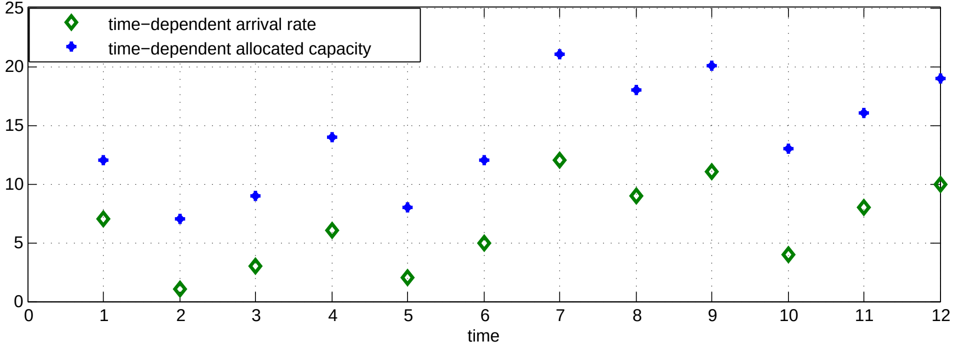


Fig.4 A dimensioning example of a single link with nonstationary arrival rates



Outline

Why try different solutions?

1. We try to compare and find efficient approaches
2. Our goal is to dimension a communication system, calculating the blocking probabilities is the first step.
3. Q: Which solution is good for dimensioning?

Stationary Arrivals

From the Continuous Time Markov Chain of M/M/s/s queue, we obtain

$$
Q = \begin{bmatrix}
-\lambda & \lambda & 0 & & \cdots & 0 \\
\mu & -\mu-\lambda & \lambda & & \cdots & 0 \\
0 & 2\mu & -2\mu-\lambda & & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \cdots & & \lambda & 0 \\
0 & \cdots & (s-1)\mu & & -(s-1)\mu-\lambda & \lambda \\
0 & \cdots & 0 & & s\mu & -s\mu
\end{bmatrix} \tag{7}
$$

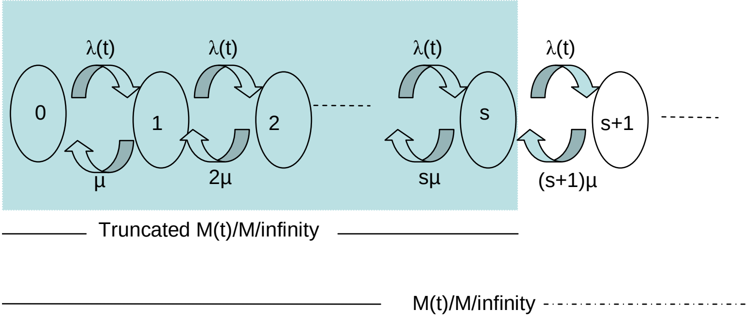
$$
P_s(t) = \beta e^{(Q^T)t}\alpha \tag{8}
$$

where α is the initial state probability vector , β is an all-zero row vector except that the last entry is 1.

Nonstationary Arrivals

An explicit solution is provided in Jagerman [6] by using the probability generating functions of the state probabilities and the corresponding binomial moments where arrival rate function $\lambda(t)$ is considered to be continuous.
However, it's too complex to use for practice.
Q: What to do?

The Truncated Markov Process–An Approximation



Stationary Arrivals

we can obtain the transient distribution of $M/M/\infty$ [14]:

$$P_n^{\infty}(t) = \frac{m^n e^{-m}}{n!}, m \equiv m(t) = \rho(1 - e^{-\mu t}). \quad (9)$$

applying the truncation property approximately for an M/M/s/s queue [6]:

$$P_n^s(t) \approx \frac{P_n^{\infty}(t)}{\sum_k P_k^{\infty}(t)}, k \in \{0, 1, ...s\} \quad (10)$$

Nonstationary Arrivals

For the $M(t)/M/\infty$ queue assuming service rate $\mu = 1$:

$$P_n^\infty(t) = \frac{\rho(t)^n e^{-\rho(t)}}{n!}, \rho(t) = e^{-t}\int_0^t \lambda(u)e^u du \qquad (11)$$

applying truncated property to M(t)/M/s/s system [6]:

$$P_n^\infty(t) \approx P\{q_\infty(t) = n | q_\infty(t) < s\} = \frac{\rho(t)^n/n!}{\sum_{i=0}^{s} \rho(t)^i/i!} \qquad (12)$$

Q: How accurate?

The Fixed Point Approximation (FPA) Method

The main idea of the FPA method is as follows (see Alnowibet and Perros [3]):
Given a loss queue M(t)/M/s/s with time-dependent rate $\lambda(t)$, the time-dependent average number of customers $E[Q(t)]$ can be expressed as

$$E'[Q(t)] = \lambda(t)(1 - B_p(t)) - \mu E[Q(t)] \tag{13}$$

The time-dependent mean number of customers is given by :
$E[Q(t)]$=$\rho(t)$/ $(1 - B_p(t))$, from which we can get the offered load $\rho(t)$:

$$\rho(t) = E[Q(t)]/(1 - B_p(t)). \tag{14}$$

where

$$B_p(t) = \frac{\rho(t)^s/s!}{\sum_{i=0}^{s} \rho(t)^i/i!} \tag{15}$$

The Large Deviation Approach

1. The Large Deviation theory has similarity with the Central Limit theory (CLT).
2. The CLT governs random fluctuations only near the mean–deviations from the mean of the order of $\delta/\sqrt{n}$, where δ is the standard deviation. Fluctuations which are of the order of δ are, relative to typical fluctuations, much bigger: they are large deviations from the mean.
3. An interesting asymptotic regime is obtained by scaling the arrival process (by replacing $\lambda(t)$ with $n\lambda_1(t)$).
4. Applying Cramer's theorem and Chernoff's formula: $P[\frac{1}{n}\sum_{i=1}^{n} X_i > x] \approx e^{-nI(x)}$, and $I(x) = sup_\theta\{x\theta - logM(\theta)\}$ where $M(\theta)$ represents the moment generating function of random variables $\{X_1, X_2, ...\}$.

Fig.5 The comparison among exact solution, truncated approximation (TR), Bahadur-Rao approximations(BR) and large deviations approximations (LD) where $\lambda_1(t) = 0.7 + 0.2sin(2\pi t)$, t=4.8, varying n.

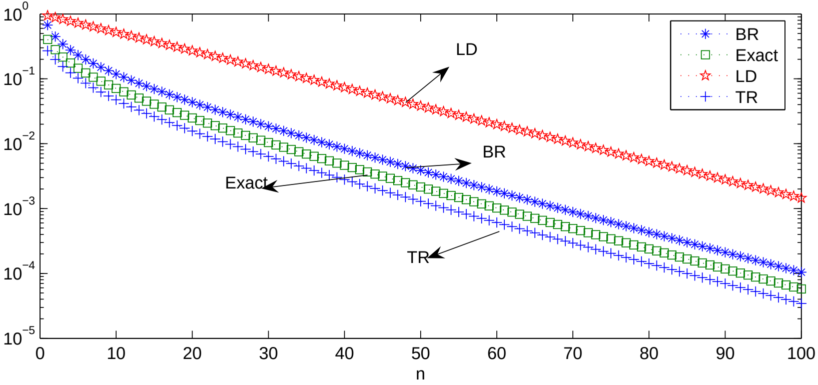
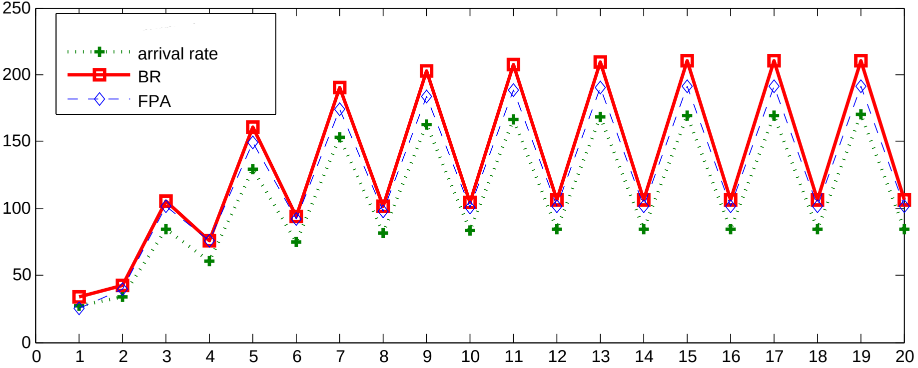


Fig.6 A dimensioning example of a single link with nonstationary arrival rates



Outline

Conclusion

Observations

- For stationary arrivals, the exact closed-form solution will be CPU efficient and easy to implement.
- For nonstationary arrivals, FPA and Bahadur-Rao approximation will be better choice.
- Bahadur-Rao approach is preferred if the system is large
- For dimensioning purpose: Which method is the best?