Performance Analysis of Assembly Systems

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Presentation Outline

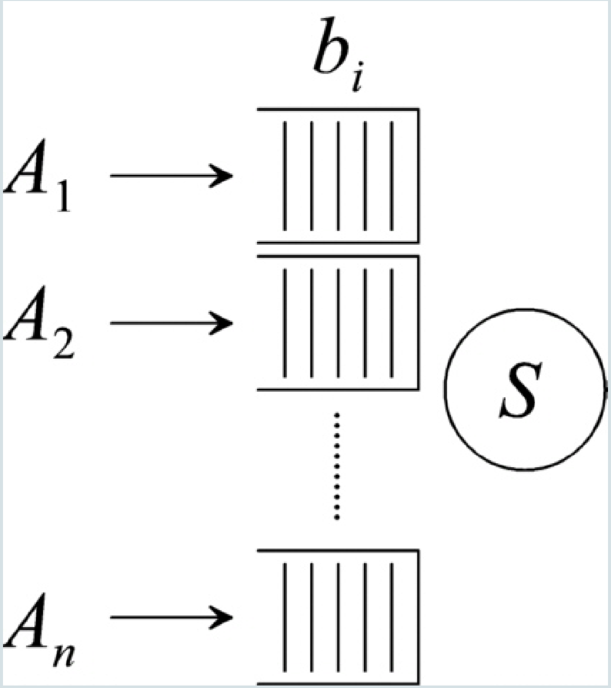
- Assembly system
- Literature
- The analysis
  - Decomposition
  - Subsystem analysis
  - Iterative algorithm
- Numerical results and conclusions

An assembly station

Model:

- General arrival processes
- Finite buffers
- Blocking after service
- General service process
- All components have to be available before service starts
- A component can wait in the server for other components

Model description



S: Service time of assembly server

A_i: Arrival process at buffer i

b_i: Buffer size of buffer i

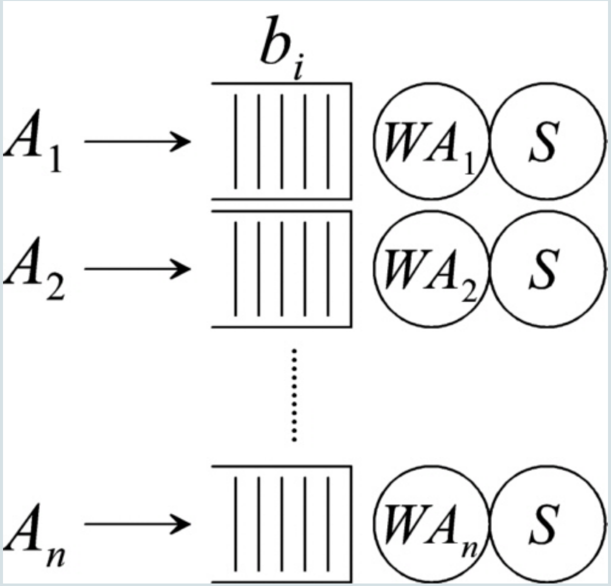
Literature

- Fork-join queue in an open network (Hemachandra and Eedupuganti)
- Fork-join queue in a closed network (Rao and Suri, and Krishnamurti et al.)
- An exact analysis of system with two parts (Gold)

Analysis approach

- Decomposition
- Subsystem Analysis
- Iterative Algorithm

Decomposition in subsystems



WA_i: Wait to assembly at buffer i

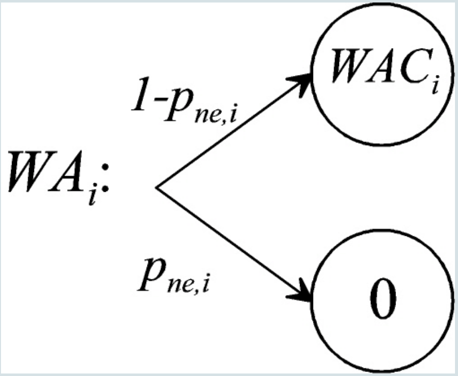
Wait to assembly at buffer i

The wait to assembly time consists of the waiting time time for the other components.
So,
$WA_i = \max_{j \neq i} RA_j$

Note:

- The RA_j's can be obtained from the subsystems
- Both the WA's and RA's have mass in zero

Wait to assembly at buffer i



$$\max_{1\le j\le k} RA_j = \max\{RA_k, \max_{1\le j\le k-1} RA_j\}$$
$$P(\max_{1\le j\le k-1} RA_j = 0) = \prod_{1\le j\le k-1} (1 - p_{e,j})$$
$$p_{ne,i} = \prod_{j\ne i} (1 - p_{e,j})$$

Maximum of two random variables (1)

E_1 and E_2 Erlang distributed with parameters k_i and μ_i $(i = 1, 2)$
First a number op phases with parameter $\mu_1 + \mu_2$ then either μ_1 or μ_2

$q_{k,j}$: k wins and the other finished j phases

$$q_{1,j} = \binom{k_1 - 1 + j}{k_1 - 1} \left(\frac{\mu_2}{\mu_1 + \mu_2}\right)^j \left(\frac{\mu_1}{\mu_1 + \mu_2}\right)^{k_1}, \quad 0 \leq j \leq k_2 - 1$$

$$q_{2,i} = \binom{k_2 - 1 + i}{k_2 - 1} \left(\frac{\mu_1}{\mu_1 + \mu_2}\right)^i \left(\frac{\mu_2}{\mu_1 + \mu_2}\right)^{k_2}, \quad 0 \leq i \leq k_1 - 1$$

Maximum of two random variables (2)

$$\mathbb{E}M_{1,j} = \frac{k_1 + j}{\mu_1 + \mu_2} + \frac{k_2 - j}{\mu_2}$$

$$\mathbb{E}M_{1,j}^2 = \frac{(k_1 + j)(k_1 + j + 1)}{(\mu_1 + \mu_2)^2} + \frac{(k_1 + j)(k_2 - j)}{(\mu_1 + \mu_2)\mu_2} + \frac{(k_2 - j)(k_2 - j + 1)}{\mu_2^2}$$

$$\mathbb{E}(\max\{E_1, E_2\}) = \sum_{j=0}^{k_2-1} q_{1,j}\mathbb{E}M_{1,j} + \sum_{i=0}^{k_1-1} q_{2,i}\mathbb{E}M_{2,i}$$

$$\mathbb{E}(\max\{E_1, E_2\}^2) = \sum_{j=0}^{k_2-1} q_{1,j}\mathbb{E}M_{1,j}^2 + \sum_{i=0}^{k_1-1} q_{2,i}\mathbb{E}M_{2,i}^2$$

Subsystem analysis

- Fit a $E_{k-1,k}$ or C_2 distribution on the first two moments of A_i, S and WA_i
- Construct MAP's of the arrival and departure processes
- Construct a QBD of the subsystem
- Analyse the QBD by using matrix analytic methods

Constructing the QBD (1)

$$\mathbf{Q} = \begin{pmatrix} B_{00} & B_{01} & & & & \\ B_{10} & A_1 & A_0 & & & \\ & A_2 & \ddots & \ddots & & \\ & & \ddots & \ddots & A_0 & \\ & & & A_2 & A_1 & C_{10} \\ & & & & C_{01} & C_{00} \end{pmatrix}$$

Constructing the QBD (2)

Construct a MAP of the arrival process:
AR_0 and AR_1

Construct a MAP of the departure process (WA and S):
DE_0 and DE_1

Construct a MAP of the departure process in level 0 (WA):
$\widetilde{DE}_0$

Constructing the QBD (3)

$$
\begin{aligned}
A_0 &= AR_1 \otimes I_{n_{wac}+n_{sa}} \\
A_1 &= AR_0 \otimes I_{n_{wac}+n_{sa}} + I_{n_a} \otimes DE_0 \\
A_2 &= I_{n_a} \otimes DE_1 \\
B_{01} &= AR_1 \otimes I_{n_{wac}+n_{sa}} \\
B_{00} &= AR_0 \otimes I_{n_{wac}+n_{sa}} + I_{n_a} \otimes \widetilde{DE}_0 \\
B_{10} &= I_{n_a} \otimes DE_1 \\
C_{01} &= AR_1 \otimes I_{n_{wac}+n_{sa}} \\
C_{00} &= I_{n_a} \otimes DE_0 \\
C_{10} &= I_{n_a} \otimes DE_1
\end{aligned}
$$

Analyzing the QBD

$$\pi_i = x_1 R^{i-1} + x_b \hat{R}^{b-i}, \qquad i = 1, \ldots, b \tag{1}$$

$$\begin{aligned}
0 &= A_0 + RA_1 + R^2 A_2 \\
0 &= A_2 + \hat{R}A_1 + \hat{R}^2 A_0 \\
0 &= \pi_0 B_{00} + \pi_1 B_{10} \\
0 &= \pi_0 B_{01} + \pi_1 A_1 + \pi_2 A_2 \\
0 &= \pi_{b-1} A_0 + \pi_b A_1 + \pi_{b+1} C_{01} \\
0 &= \pi_b C_{10} + \pi_{b+1} C_{00}
\end{aligned}$$

Characteristics of RA

p_e: The probability that the queue is empty on departure

$$p_e = \frac{\pi_1 B_{10} e}{T},$$

α: The distribution of the phase of the inter-arrival time A just after a departure to level 0

$$\alpha = \frac{\pi_1 B_{10}}{\pi_1 B_{10} e},$$

Iterative algorithm

1. Choose initial characteristics for the wait to assembly times at each buffer
2. For each subsystem (from subsystem 1 to n):
   - Determine the wait to assembly time at buffer i
   - Determine the queue-length probabilities of the subsystem
   - Determine a new RA_i

Repeat step 2 until the characteristics of the WA's of the subsystems have converged.

Numerical results

The following parameters are varied:

- number of parts: 2, 4, 8
- buffersize: 0, 2, 4, 8
- SCV of the arrivals: 0.2, 0.5, 1, 2
- SCV of the service process: 0.5, 1
- Occupation rate: 0.75, 1
- Imbalance in the arrival rates
- Imbalance in the SCV's of the arrivals

A total of 768 test cases.

Numerical results (2)

Perf. char.	Avg.	0-5 %	5-10 %	$>$ 10 %
Throughput	1.5 %	97.4 %	2.6 %	0.0 %
Mean sojourn time	2.8 %	84.9 %	13.4 %	1.7 %

Most sensitive for:

- different buffer sizes
- different number of parts

Conclusions

Conclusions:

- Very good results
- Fast computation
- A good method for analyzing assembly systems

Future research:

- Incorporate the method in a network setting